PROBABILISTIC DAMAGE TOLERANCE ANALYSIS USING ADAPTIVE MULTIPLE IMPORTANCE SAMPLING

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Abstract: This paper discusses the importance of Probabilistic Damage Tolerance Analysis (PDTA) tools in ensuring commercial and military fleets' continued operational safety (COS). The traditional approach to calculating single-flight-probability-of-failure (SFPOF) has limitations and assumptions that can affect the confidence in the estimate. Under Federal Aviation Administration (FAA) sponsorship, our team has been developing a risk assessment computer code, SMART|DT, that can account for important random variables such as material properties, loading usage, inspection probability of detection, and build and repair quality. The focus of this paper is the Adaptive Multiple Importance Sampling (AMIS) method, which provides 5 to 6 orders of magnitude improvement in computational efficiency for comprehensive PDTA compared to standard Monte Carlo sampling. AMIS enables the use of realistic fracture mechanics models and a large number of random variables to be considered. The AMIS method can be applied not only to the COS of aircraft fleets but also to digital twin modeling, virtual testing, and other new applications. Two real-case scenario examples demonstrate the accuracy and efficiency of the AMIS method using a comprehensive set of random variables for management of aircraft fleets.

Keywords: Probabilistic Damage Tolerance, Adaptive Multiple Importance Sampling.

INTRODUCTION

The continued operational safety (COS) of commercial and military fleets relies on Probabilistic Damage Tolerance Analysis (PDTA) tools to effectively assess and manage the risk of structural failure. PDTA enables risk assessment and management by calculating the single-flight-probability-of-failure (SFPOF) as a function of flight hours. The SFPOF of an aircraft component is challenging to compute due to its small probability, typically 10^{-7} or less. Traditionally, it is calculated with limitations on the number of random variables and assumptions on fracture mechanics that may affect the confidence in the SFPOF estimate. Furthermore, these limitations and assumptions inhibit the use of the latest developments in fracture mechanics modeling, structural health monitoring, material modeling, and manufacturing due to the absence of efficient probabilistic methods to successfully calculate the risk.

Under Federal Aviation Administration (FAA) sponsorship, our team has been developing a risk assessment computer code, SMART|DT, for aircraft structures that can account for the variability of important parameters such as material properties, usage, inspection probability of detection, and build quality. This presentation will focus on an Adaptive Multiple Importance Sampling (AMIS) method that provides 5 to 6 orders of magnitude improvement in computational efficiency for performing comprehensive PDTA compared to standard Monte Carlos sampling. The most fundamental aspect of AMIS is that it will detect the important values for each variable that contribute the most to the SFPOF. In addition, since the probability-of-failure is needed at multiple flight hours to assess risk, a mixture density consisting of a weighted combination of many component densities is developed which can accurately estimate SFPOF across all analysis times requested by the user. The AMIS method allows one to consider more realistic fracture mechanics models and a larger number of random variables than has been previously possible. AMIS can be used not only for the COS of aircraft fleets, but also for applications such as digital twin modeling, virtual testing, and other new applications.

METHODOLOGY

SMART|DT

The SMART|DT software [1, 2] employs a comprehensive methodology consisting of five main components for performing a PDTA. These components include aircraft load generation, extreme value distribution (EVD) generation, fracture mechanics crack growth module with links to NASGRO and internal crack growth capabilities [3], inspection and repair module, and probabilistic methods for generating random variables and computing the SFPOF at any point in the aircraft's lifespan. Furthermore, the software allows for fleet management and a Bayesian updating module will be implemented to use field inspection data to update the crack size distributions. Figure 1 shows the modules within SMART|DT, which can generate the necessary inputs for multiple fracture mechanics evaluations, enabling fracture mechanics parameters to be random variables within the PDTA.

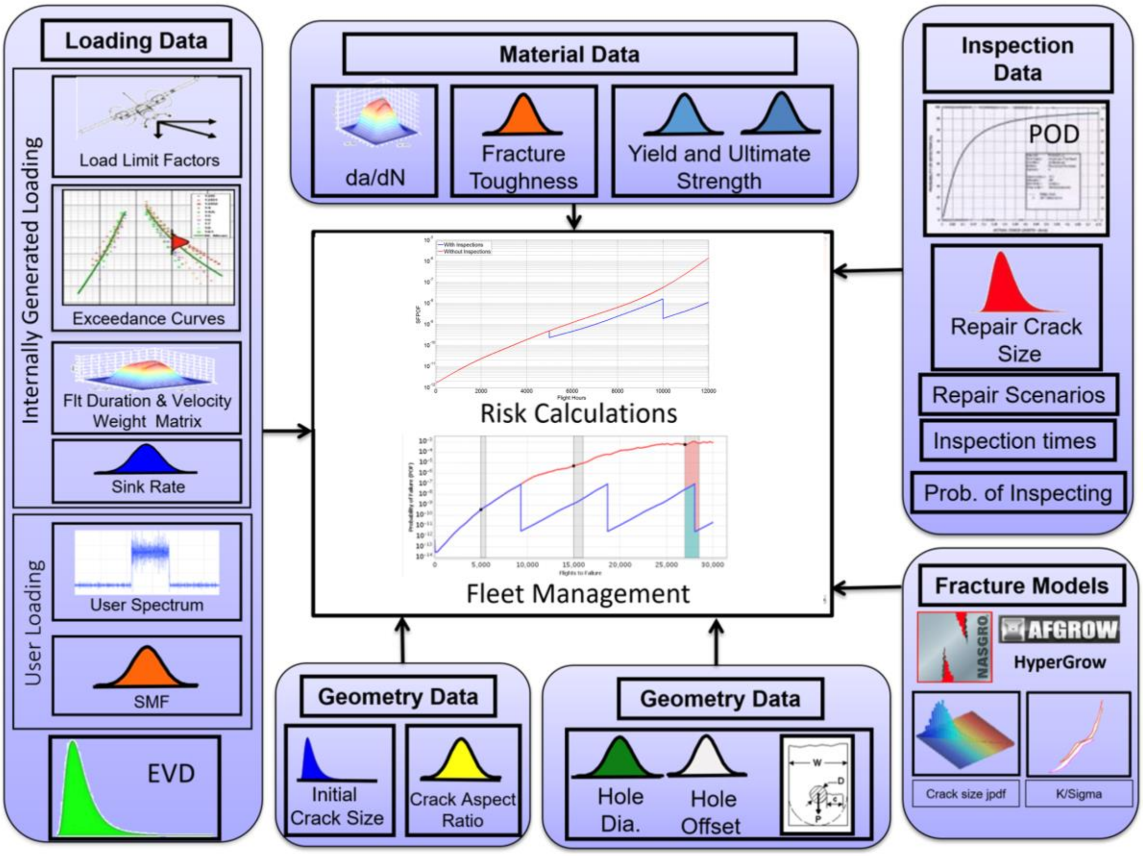

Figure 1: SMART|DT Methodology

In PDTA, the probability of failure (POF) is the probability that the maximum value of the applied stress (during the next flight) will exceed the residual strength σ_{RS} of the aircraft component and it can be written mathematically as follows:

$$POF(t) = P[\sigma_{MAX} > \sigma_{RS}(t)] \quad (1)$$

The single flight probability of failure (SFPOF) is defined as the probability of failure on the next flight assuming survival until that flight. If one assumes that the probability of survival is one over many flights for small probabilities of failure, e.g. certain survival, the conditional probability of failure is given by Eqn. 2, which is referred to as Lincoln POF in the literature. The SFPOF for a given time, t, is calculated as $SFPOF(t) = E[SFPOF_{cond}(\boldsymbol{x}, t)]$, the expected value of $SFPOF_{cond}$ over all possible values of random variables $\boldsymbol{x}$ as shown in Eqn. 3.

$$SFPOF_{cond}(\boldsymbol{x}, t) = 1 - F_{EVD}\big(\sigma_{RS}(\boldsymbol{x}, t)\big) \quad (2)$$

$$SFPOF(t) = \int SFPOF_{cond}(\boldsymbol{x}, t)\, f(\boldsymbol{x})\, d\boldsymbol{x} \quad (3)$$

If survival is not assumed to be certain for small probabilities of failure, the Freudenthal SFPOF is calculated as a hazard function, Eqn. 6, of the cumulative probability of failure up to time t over n flights. The conditional cumulative probability of failure, including a survival term which is the product of the probability of surviving all prior flights, is given in Eqn. 4, and the derivative of $CTPOF_{cond}$, derived by taking the finite difference of $CTPOF_{cond}(\boldsymbol{x}, t_n)$ and $CTPOF_{cond}(\boldsymbol{x}, t_n - 1)$ with respect to n, is shown in Eqn. 5. The resulting hazard function is then calculated from the expected values of $CTPOF_{cond}$ and its derivative, $Hz(CTPOF_{cond}) = E[d\, CTPOF_{cond}(\boldsymbol{x}, t)/\mathrm{d}t]/(1 - E[CTPOF_{cond}(\boldsymbol{x}, t)])$.

$$CTPOF_{cond}(\boldsymbol{x}, t) = 1 - \prod_{i=1}^{n} F_{EVD}\big(\sigma_{RS}(\boldsymbol{x}, t_i)\big) \quad (4)$$

$$d\, CTPOF_{cond}(\boldsymbol{x}, t)/\mathrm{d}t = \left[\prod_{i=1}^{n-1} F_{EVD}\big(\sigma_{RS}(\boldsymbol{x}, t_i)\big)\right]\Big(1 - F_{EVD}\big(\sigma_{RS}(\boldsymbol{x}, t_n)\big)\Big) \quad (5)$$

$$SFPOF(t) = \frac{\int \mathrm{d}\, CTPOF_{cond}(\boldsymbol{x}, t)/\mathrm{d}t\;\, f(\boldsymbol{x})\, d\boldsymbol{x}}{1 - \int CTPOF_{cond}(\boldsymbol{x}, t)\;\, f(\boldsymbol{x})\, d\boldsymbol{x}} \quad (6)$$

Failure can occur by unstable fracture, $K_I \geq K_C$, where K_I denotes the stress intensity factor and K_C is the fracture toughness, or net section yield (NSY), or the crack reaching the width of the part.

The SFPOF above is challenging to compute due to its small probability, typically 10^{-7} or less. When variation of the crack growth parameters and additional random variables are included in PDTA, the computational cost becomes prohibitive for standard Monte Carlo (SMC) and quadrature integration methods. For this reason, this paper will discuss the implementation of the AMIS algorithm that provides 5 to 6 orders of magnitude improvement in computational efficiency for performing comprehensive PDTA.

Adaptive Multiple Importance Sampling (AMIS)

SMC generates realizations from the nominal joint distribution probability density function, $f(\boldsymbol{x})$, and calculates a sampling approximation, Eqn. 7, of the integral functions in Eqns. 3 and 6 as a sum of the conditional values divided by the number of realizations where any of the conditional equations defined in the previous section, for example in Eqn. 2 for Lincoln, can be substituted for $H(\boldsymbol{x}, t)$ and N_{samp} is the number of realizations.

$$E[H(x, t)] = \frac{1}{N_{samp}} \sum_{i=1}^{N_{samp}} H(\boldsymbol{x}_i, t) \quad (7)$$

Figure 2 depicts the difficulty of estimating small probabilities with SMC. On average, only one in ten realizations will be outside the inner-most ellipse, and the proportion falls an order of magnitude with each ellipse, down to one in ten million for the outer-most ellipse. The failure region, shaded grey, represents where the conditional probability of failure is greater than 0.999. The important region, shaded red, shows the extent of the region bounding 99% of the integrated SFPOF (shaded gray). With SMC, a very large sample size of approximately 10^9 is required to generate enough realizations in the important region for a good estimate.

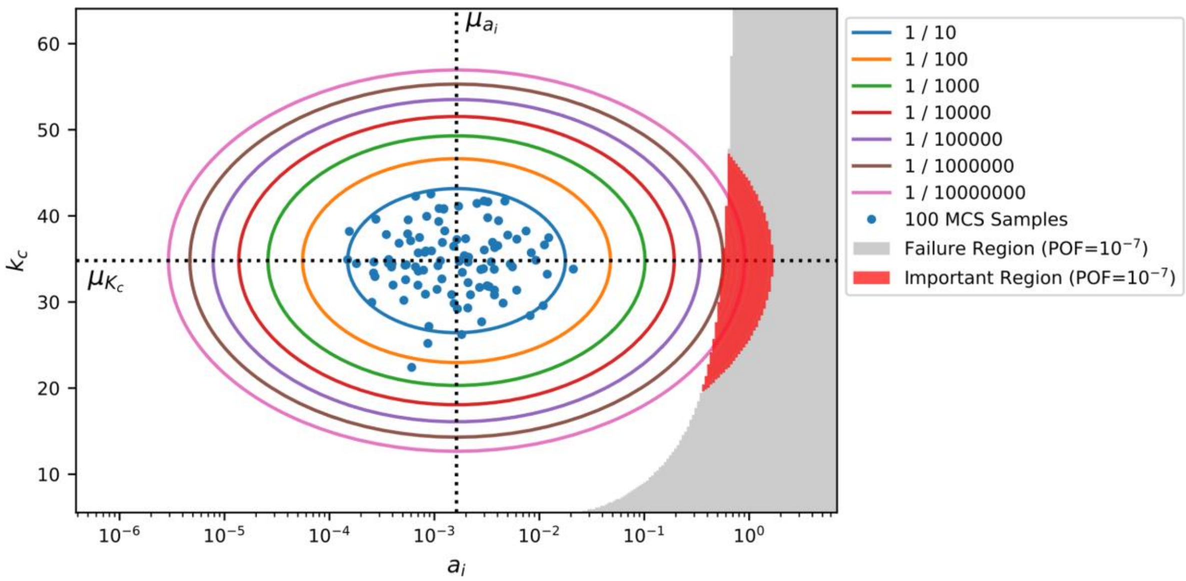

Figure 2: Standard Monte Carlo Sampling

Importance sampling (IS) provides a much more efficient method to compute the SFPOF than SMC. IS allows the realizations to be generated from a sampling density $g(\boldsymbol{x})$ and corrects for the sampling bias by weighting each sample by the likelihood ratio, $f(\boldsymbol{x})/g(\boldsymbol{x})$, which is also referred to as the importance weight, $w(\boldsymbol{x})$ as shown in Eqn. 8.

$$E[H(x,t)] = \frac{1}{N_{samp}} \sum_{i=1}^{N_{samp}} H(\boldsymbol{x}_i, t) \frac{f(\boldsymbol{x}_i)}{g(\boldsymbol{x}_i)} = \frac{1}{N_{samp}} \sum_{i=1}^{N_{samp}} H(\boldsymbol{x}_i, t)\, w(\boldsymbol{x}_i) \tag{8}$$

Importance sampling can generate nearly all of the samples in the important region if the important region location and shape are known. The difficulty with importance sampling is that samples outside of the important region can become infinitely weighted if the tails of $g(\boldsymbol{x})$ decay faster than the tails of the product of the conditional probability and nominal density, $H(\boldsymbol{x}, t)\, f(\boldsymbol{x})$. Hence, the sampling density must be selected carefully because a poor selection can increase the variance of the importance sampling estimate.

Adaptive importance sampling methods learn the location and shape of the important region by repeatedly taking a small sample and adjusting the sampling density. Many adaptive importance sampling methods are based on adapting a single multivariate importance sampling density. This limits reuse of samples evaluated in the adaptation process because including samples from any density other than the current $g(\boldsymbol{x})$ will bias the final result.

Mixture densities allow multiple sampling densities to be combined and used as the importance sampling density as shown in Eqn. 9 where $\boldsymbol{x}_{ij}$ are N_{samp} realizations generated from each of N_{mix} component densities, $g(\cdot, \theta_j)$, and ω_j is are weighting functions that define a partition of unity, requiring

$\sum_{m=1}^{N_{mix}} \omega_m(\boldsymbol{x}_{ij}) = 1$ for all $\boldsymbol{x}_{ij}$. The notation for component densities in Eqn. 9 changes signifying that a parametric distribution is used for the component densities with distribution parameters $\boldsymbol{\theta}_j$ for the j-th component density.

$$E[H(x,t)] = \sum_{j=1}^{N_{mix}} \frac{1}{N_{samp}} \sum_{i=1}^{N_{samp}} H(\boldsymbol{x}_{ij}, t)\, \omega_j(\boldsymbol{x}_{ij})\, \frac{f(\boldsymbol{x}_{ij})}{g(\boldsymbol{x}_{ij}, \theta_j)} \tag{9}$$

With standard mixture weighting, Eqn. 10, the likelihood ratio of samples drawn from the tail of the sampling density can still be magnified by several orders of magnitude, and because standard mixture weights only depend on $g(\cdot, \theta_j)$ from the component density that generates samples $\boldsymbol{x}_{ij}$, badly weighted samples continue to affect the final expectation value and variance until a large number of samples are accumulated to correct the balance. Importance sampling with mixture densities was not widely adopted until other methods of mixture weighting were developed.

$$\omega_m^{std}(\boldsymbol{x}_{ij}) = \begin{cases} 1, & m = j \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

Veach and Guibas conceived that as long as the weighting functions define a partition of unity, the resulting mixture importance sampling estimator would remain unbiased, and proposed several weighting functions[4]. In addition to standard mixture weighting, Veach and Guibas defined what has since been referred to as balance heuristic mixture weighting shown in Eqn. 11. When an equal number of samples are generated from each component density, $n_k = n_m = N_{samp}$, Eqn. 12 results from substituting Eqn. 11 into Eqn. 9 and simplifying. From Eqn. 12, it can be intuitively understood that very small values resulting from samples generated in the tail of any of the sampling densities will no longer have a oversized impact on the final estimate and estimate variance because when summed over all of the component densities, values that are orders of magnitude larger dominate.

$$\omega_m^{bh}(\boldsymbol{x}_{ij}) = \frac{n_m\, q(\boldsymbol{x}_{ij}, \boldsymbol{\theta}_m)}{\sum_{k=1}^{N_{mix}} n_k\, q(\boldsymbol{x}_{ij}, \boldsymbol{\theta}_k)} \tag{11}$$

$$E[H(x,t)] = \frac{1}{N_{mix}\, N_{samp}} \sum_{j=1}^{N_{mix}} \sum_{i=1}^{N_{samp}} H(\boldsymbol{x}_{ij}, t)\, \frac{f(\boldsymbol{x}_{ij})}{(1/N_{mix}) \sum_{k=1}^{N_{mix}} q(\boldsymbol{x}_{ij}, \boldsymbol{\theta}_k)} \tag{12}$$

Owen and Zhou further showed that an upper bound is placed on the importance weights for all samples when balance heuristic weights are used with a mixture that includes the nominal density [5]. Cornuet et al. developed an adaptive multiple importance sampling method using balance heuristic importance weights and applied the method to several academic example problems [6]. Thijssen and Kappen showed that using standard weights during the adaptation process to determine new component density parameters addressed consistency and convergence issues with the original AMIS algorithm [7].

Using balance heuristic weighting with a mixture importance sampling density which includes the nominal density as one of the component densities enables the resulting importance sampling algorithm to collectively use all of the samples generated in each adaptation trial instead of throwing away all but the final set. In addition, because balance heuristic weighting promotes the most favorable likelihood ratio from all of the component densities due to the summation term in the denominator, realizations from every component density contribute with low variance to the final estimate once the near optimal sampling density is found.

The AMIS algorithm presented in this paper adapts a mixture importance sampling density consisting of many component multivariate normal densities to estimate SFPOF values over a range of evaluation times [8]. Sampling is conducted in standard normal space which allows a common scaling to be used across all random variables. The random numbers for each variable are transformed back from standard normal to physical space before evaluating the crack growth function.

The initial sampling density is centered at the origin in standard normal space with an identity covariance matrix multiplied by a scaling factor of 9 corresponding to a standard deviation of 3 for each random variable. This initial sampling satisfies the conditions to assure an upper bound for all balance heuristic importance weights. The algorithm then explores the parameter space by adding new multivariate normal component densities with the same scaled covariance matrix for a subset of the SFPOF evaluation times such that for each evaluation time, the calculated optimal sampling density location is within a preset distance of one of the initialization component densities using a relative entropy metric. After adding initialization component densities that cover the important regions for all SFPOF evaluation times, additional component densities are adaptively added to the mixture for the evaluation time with the highest coefficient of variation (COV) until the SFPOF estimate COVs for all evaluation times are all below the target COV threshold. Once the adaptation process finishes, the final SFPOF values are calculated using Eqn. 12 where the conditional probability of failure functions from the previous section are substituted for $H(\boldsymbol{x}, t)$ as needed.

Figure 3 presents an instance of traditional importance sampling for two random variables, the initial crack size (a_i) and fracture toughness (k_c), where the probability density changes over time, as shown on the left-hand side. The right-hand side illustrates the density mixture for the same two random variables that were generated utilizing the AMIS algorithm.

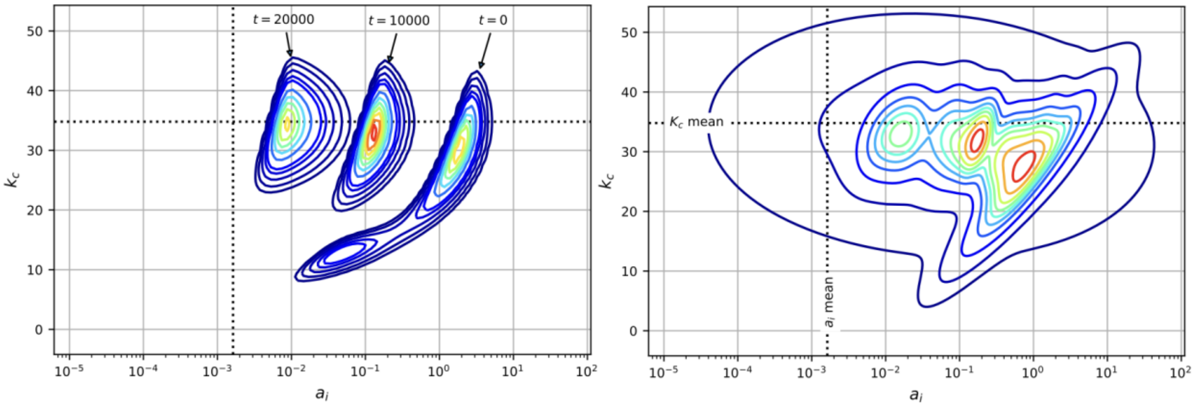

Figure 3: Traditional Importance Sampling (left) and Adaptive Multiple Importance Sampling (right)

Two example problems are presented to demonstrate the advantages of the AMIS method to improve the computational efficiency for performing comprehensive PDTA. The method is available within the SMART|DT software that can be accessed by contacting the authors.

EXAMPLE PROBLEMS

Example 1 – Handbook Example Problem

The handbook example problem involves a through crack that propagates from a fastener hole positioned at the center of a plate under remote tensile loading. Figure 4 provides a schematic of the problem.

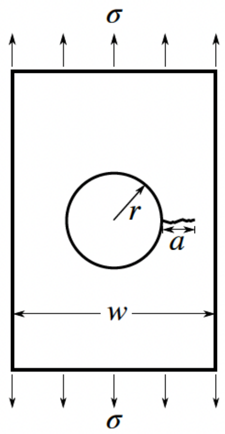

Figure 4: Through Thickness Crack in a Fastener Hole.

The handbook problem uses a closed-form solution for the crack size versus time, as shown in Eqn. 13, which allows for a much faster calculation but gives up the capability to model crack growth as a function of remote stress and crack growth rate parameters. The residual stress shown in Eqn. 15 can be calculated directly from the crack size, fracture toughness, and the geometry factor given in Eqn. 14, which is a product of geometry factors for a crack growing from a hole and crack approaching an edge.

$$a(t) = a_0 \exp(t \cdot 2.93 \times 10^{-4}) \tag{13}$$

In Eqn. 14 the first half of the right-hand side is the beta factor for the hole and the second half is the beta width factor.

$$\beta(a(t)) = \left(0.6762 + \frac{0.8734}{0.3246 + a(t)/r}\right) \times \left(\sqrt{sec\left(\frac{\pi(r+a(t))}{w}\right)}\right) \tag{14}$$

$$\sigma_{rs}(a(t)) = \frac{K_c}{\beta(a(t))\sqrt{\pi a(t)}} \tag{15}$$

Table 1 defines the variables for the handbook problem. The probability distribution and distribution parameters are specified also in Table 1. Deterministic values are given for the width and hole radius. The Weibull minimum distribution is used for the maximum stress per flight, σ_{EVD}, in this example. The CDF of the three-parameter Weibull minimum distribution is $F(x) = 1 - exp(-((x-\mu)/\beta)^{\alpha})$ with location μ, scale β, and shape α.

Table 1: Handbook problem variable definitions.

Random Variable	*Distribution*	*Parameters*
Initial crack size, $\boldsymbol{a_0}$	Lognormal	mean = 0.003 in standard deviation = 0.0047 in
Fracture toughness, K_C	Normal	mean = 34.8 ksi· in$^{1/2}$ standard deviation = 3.9 ksi· in$^{1/2}$
Maximum stress, σ_{EVD}	Weibull	Location = 5.0 ksi Scale = 10.0 ksi Shape = 5.0
Width, w		10.0 in
Hole radius, r		0.125 in

Liao's results [9] using the NRC PDTA software ProDTA have been digitized and are reproduced here along with results from the SMART|DT AMIS algorithm. Figure 5 displays the AMIS PDTA and

ProDTA SFPOF results in the top plot, while the AMIS PDTA estimator COV is shown in the bottom panel.

For the Lincoln SFPOF, the AMIS PDTA algorithm estimated SFPOF values ranging from 10^{-10} to 10^{-4}, with 10% COV for 15 evaluation times ranging from *t = 0* throughout *t = 10,000*, using a total of 880 realizations. For the Fruedenthal SFPOF the AMIS algorithm estimated SFPOF values ranging from 10^{-11} to10^{-7}, with 10% COV at 15 evaluation times, using a total of 3040 realizations.

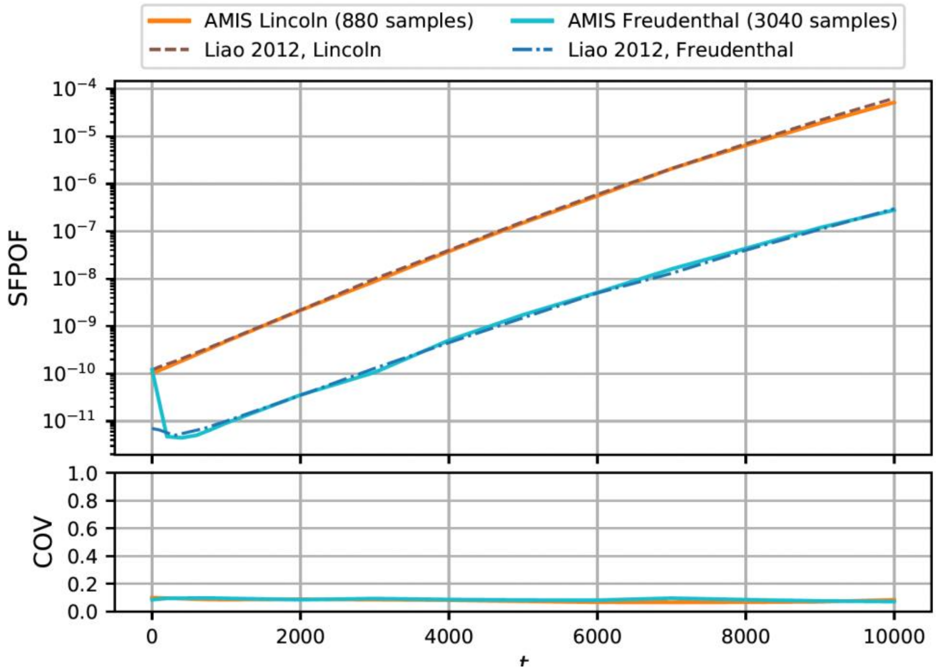

Figure 5: Top: AMIS PDTA SFPOF Lincoln formulation (orange) and Freudenthal formulation (cyan), ProDTA SFPOF Lincoln formulation (brown dashed) and Freudenthal formulation (blue dash-dot). Bottom: AMIS PDTA COV for Lincoln formulation (orange) and Freudenthal formulation (cyan) SFPOF estimators

<u>Example 2 – Capstone Example Problem</u>

This example problem introduces additional random variables and the crack growth analyses are evaluated using NASGRO [10]. The geometry for this example is a corner crack growing from a fastener hole under remote tension. Crack growth is evaluated by NASGRO using an equivalent constant amplitude stress spectrum [11]. Table 2 gives the random variable definitions for the problem which includes 5 random variables. The inspection schedule starts at 8,000 flights and the inspection and repair process keeps the risk below 10^{-7}.

Table 2: Capstone problem random variable definitions.

Random Variable	*Distribution*	*Parameters*
Initial crack size, $\boldsymbol{a_0}$	Weibull	scale = 4.17 x 10^{-5} in shape = 0.45 in
Fracture toughness, K_C	Normal	mean = 35.0 ksi· $in^{1/2}$ standard deviation = 3.5 ksi· $in^{1/2}$
Log_{10} Paris C	Normal	mean = -9.0 standard deviation = 0.08
Maximum stress, σ_{EVD}	Weibull	Location = 5.0 ksi Scale = 1.3 ksi Shape = 0.007
Probability of Detection POD	Lognormal	mean = 0.0215 in standard deviation = 0.05 in
Repair crack size	Lognormal	mean = 0.065 in standard deviation = 0.004 in
Paris Exponent m		3.8
Width, w		5.0 in
Hole radius, r		0.125 in

Figure 6 presents a comparison of the SFPOF results obtained using the AMIS algorithm and those obtained from 10^9 SMC samples. The figure shows that the AMIS results are consistent with the SMC sampling results, indicating that the AMIS algorithm can produce accurate results when estimating probabilities across a large range of SFPOF values.

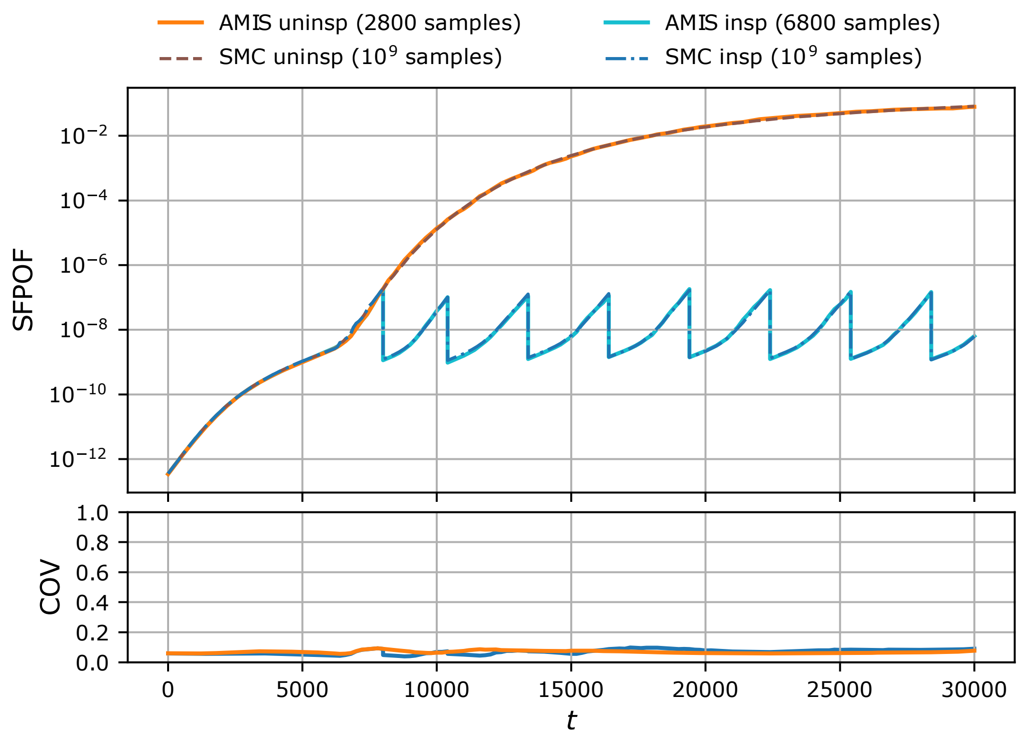

Figure 6: Top: PDTA AMIS uninspected SFPOF result (orange) and inspected SFPOF (cyan) Monte Carlo results plotted as dashed lines using 10^9 samples. Bottom: TheAMIS coefficient of variance estimators

CONCLUSIONS

The incorporation of the AMIS algorithm into the SMART|DT software has led to a significant improvement in efficiency, as demonstrated in this paper. Specifically, compared to standard Monte Calo sampling, the AMIS algorithm is 6 orders of magnitude more efficient in estimating probabilities of failure on the order of 10^{-7}. This efficiency allows for the incorporation of additional random variables into the problem and the use of more realistic fracture mechanics solutions. Furthermore, the SMART methodology in combination with AMIS is not limited to aircraft fleet COS and can also be applied to digital twin modeling, virtual testing, and other novel applications. The accuracy and efficiency of the AMIS method are demonstrated through two real-case scenario examples, which utilize a comprehensive set of random variables.

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